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AIR MINISTRY

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Summary

The effect of refraction on the height of an object calculated from radar measurements of slant range and elevation is determined, with particular reference to the reflectors carried by free balloons used for wind finding by the Meteorological Office. The true height of an object is expressed as a function of its angle of elevation as observed from a point on the ray path and numerical solutions are obtained for a particular distribution of refractive index. The effects of departures from this refractive index structure are discussed and the conclusion is reached that a set of standard corrections is probably adequate for meteorological purposes.

The fundamental relationship

The path of a ray travelling through an atmosphere whose refractive index varies with height above the earth's surface is determined by

$$\mu r \cos\phi = \text{a constant} = \mu_0 r_0 \cos\phi_0 \text{ (say)} \qquad (1)$$

where μ is the refractive index, r is the distance from the centre of curvature of the surface of the point considered, ϕ is the angle between the tangent to the ray and the tangent to a circle through the point and concentric with the earth, and μ_0, r_0 and ϕ_0 are the corresponding values where the ray intersects the surface of the earth. See Fig.1.

If the path of the ray is specified in polar co-ordinates with the centre of the earth as origin we have

$$\tan\phi = \frac{dr}{r\,d\theta}$$

Thus
$$\mu r \cos\phi = \mu r \Big/ \sqrt{1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2}$$

$$= \frac{\mu r^2 \frac{d\theta}{dr}}{\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}} = \mu_0 r_0 \cos\phi_0$$

$$\mu_0^2 r_0^2 \cos^2\phi_0 \left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right] = \mu^2 r^4 \left(\frac{d\theta}{dr}\right)^2$$

$$\therefore \quad r^2\left(\frac{d\theta}{dr}\right)^2\left[\mu^2 r^2 - \mu_0^2 r_0^2 \cos^2\phi_0\right] = \mu_0^2 r_0^2 \cos^2\phi_0$$

$$\frac{d\theta}{dr} = \frac{\mu_0 r_0 \cos\phi_0}{r\sqrt{\mu^2 r^2 - \mu_0^2 r_0^2 \cos^2\phi_0}}$$

$$\int d\theta = \int \frac{\cos\phi_0 \, dr}{r\sqrt{\left(\frac{\mu r}{\mu_0 r_0}\right)^2 - \cos^2\phi_0}} \qquad \text{(2)}$$

Before the expression on the right-hand side can be integrated it is necessary to express μ as a function of r.

Refractive index structure of the atmosphere

The radio refractive index μ at a point in the atmosphere is given by the expression

$$(\mu - 1)\,10^6 = \frac{79}{T}\left(P - \frac{e}{T} + \frac{4800\,e}{T}\right)$$

where P is the total atmospheric pressure (mb), e is the partial pressure of water vapour (mb) and T is the air temperature (°A). The refractive index can be expressed as a function of height if T and e are similarly specified.

The I.C.A.N. standard atmosphere specifies a relationship between pressure, temperature and height which is generally acceptable as representative of average conditions but no such standard has been accepted with regard to the moisture structure. Values of $(\mu-1) \times 10^6$ were therefore calculated both for the I.C.A.N. atmosphere and for an atmosphere comprising the I.C.A.N. values together with a moisture structure in which the relative humidity decreases from 80% at the surface to 10% at 300 mb and is zero thereafter. These atmospheres are referred to respectively as "ICAN dry" and "ICAN moist", and a selection of values is given in Table I.

TABLE I

Height above earth's surface (thousands of feet)		0	4	8	12	16	20	30	36.09	40	50	60	80	100
$(\mu-1) \times 10^6$	ICAN moist	342.0	282.0	236.0	200.6	172.8	149.5	102.8	81.4	67.5	41.7	25.9	9.3	3.6
	ICAN dry	278.3	247.1	218.8	192.9	169.6	148.2	102.8	81.4	67.5	41.7	25.9	9.3	3.6

Effect of an "ICAN moist" refractive structure

It has been suggested (1) that radio refractive index can best be represented

/as

as a logarithmic function of height. It was found that the equation

$$\frac{\mu r}{\mu_0 r_0} = 1 + A \log \frac{r}{r_0} + B\left(\log \frac{r}{r_0}\right)^2 \qquad (3)$$

closely represented the values of Table I providing that the atmosphere was treated in a series of layers and appropriate values of A and B chosen for each. This formula was also convenient for the integration of Equation (2). Values of the constants and of the layers for the "ICAN moist" atmosphere are given in Table II.

TABLE II

Layer (feet)	0 - 20,000	20,000-50,000	50,000-80,000	80,000-100,000
Value of A	0.672 728 676	0.885 230 233	0.964 899 698	0.992 113 907
constant B	132.072 415 0	28.035 019 5	4.642 535 83	2.490 555 066
Height	4,000 feet 15,000 feet	40,000 feet	70,000 feet	
Error $(\mu - 1) \times 10^6$	5 4	2	2	

From equation (3)

$$\left(\frac{\mu r}{\mu_0 r_0}\right)^2 = 1 + 2A \log \frac{r}{r_0} + \left(A^2 + 2B\right)\left(\log \frac{r}{r_0}\right)^2 + 2AB\left(\log \frac{r}{r_0}\right)^3 + B^2\left(\log \frac{r}{r_0}\right)^4$$

$\log \frac{r}{r_0}$ is small and terms of higher order than the squares may be neglected.

$$\therefore \left(\frac{\mu r}{\mu_0 r_0}\right)^2 = \left(\alpha \log \frac{r}{r_0} + \beta\right)^2 + 1 - \beta^2 \qquad (3(a))$$

where $\alpha = (A^2 + 2B)^{\frac{1}{2}}$

and $\beta = A(A^2 + 2B)^{-\frac{1}{2}}$

Substituting in Equation (2)

$$d\theta = \int \frac{\cos \phi_0 \, dr}{r\left\{\left(\alpha \log \frac{r}{r_0} + \beta\right)^2 + 1 - \beta^2 - \cos^2 \phi_0\right\}^{\frac{1}{2}}}$$

$$= \frac{\cos \phi_0}{\alpha} \int \left(X^2 - C^2\right)^{-\frac{1}{2}} dX$$

where $X = \left(\alpha \log \frac{r}{r_0} + \beta\right)$ and $C^2 = \beta^2 - \sin^2 \phi_0$

/Thus

Thus if $\theta = 0$ and $X = X_0$ when $r = r_0$ we have on integration

$$\frac{\alpha\theta}{\cos\phi_0} = \cosh^{-1}\frac{X}{C} - \cosh^{-1}\frac{X_0}{C}$$

$$= \cosh^{-1}\left\{\frac{\alpha \log\frac{r}{r_0} + \beta}{(\beta^2 - \sin^2\phi_0)^{\frac{1}{2}}}\right\} - \cosh^{-1}\left\{\frac{\beta}{(\beta^2 - \sin^2\phi_0)^{\frac{1}{2}}}\right\}$$

whence

$$\alpha \log\frac{r}{r_0} + \beta = (\beta^2 - \sin^2\phi_0)^{\frac{1}{2}}\left[\frac{\beta\cosh(\alpha\theta\sec\phi_0)}{(\beta^2 - \sin^2\phi_0)^{\frac{1}{2}}} + \sinh(\alpha\theta\sec\phi_0)\left(\frac{\beta^2}{\beta^2 - \sin^2\phi_0} - 1\right)^{\frac{1}{2}}\right]$$

$$= \beta\cosh(\alpha\theta\sec\phi_0) + \sin\phi_0\sinh(\alpha\theta\sec\phi_0) \qquad \text{(4)}$$

Values of r were then obtained from Equation (4) for a series of values of ϕ_0 and θ. The calculations had to be carried out in steps corresponding to the layers in Table II but the ϕ_0 for each layer were calculated from Equation (1) to correspond with the series of ϕ_0 at the surface used in the lowest layer. Thus values of r and θ were obtained for each of a series of rays arriving at the surface at selected angles of elevation.

The quantity $r - r_0 = h$ is the true height of a target which would be observed by radar to have an angle of elevation ϕ_0 and a slant range of say R. The height of the target for routine work is computed with allowance for curvature of the earth from the formula:

$$h' = R\sin\phi_0 + \frac{(R\cos\phi_0)^2}{2r_0} \qquad \text{(5)}$$

In order to compare the apparent height (h') with the true height (h) it was necessary to find R. The observed slant range is in fact half the time taken for a pulse to travel from the radar to the target and back multiplied by an assumed velocity of propagation, say V. Thus

$$\frac{R}{V} = \int_0^S \frac{\mu}{v_0}\, ds$$

where v_0 is the velocity of electromagnetic waves in vacuo and S is the distance measured along the path of the ray to the target. It can be demonstrated that the measured slant range does not differ significantly, for the present purpose, from the straight line distance (R') between the radar equipment and the target. Advantage has been taken of this in order to reduce the amount of calculation. Values of R' were computed for the same series of values of r and θ as obtained from Equation (4). Using these values of R' in Equation (5) corrections were obtained which could be applied to the apparent heights (h') to give true heights (h). Graphs of these corrections are given in Fig.2, which is set out in a form convenient for wind-finding work.

Departures from "ICAN moist" conditions

1. A dry atmosphere

Since this computation was carried out for a standard atmosphere it was necessary to ascertain the effect of departures from this condition. Table III compares for various elevation angles the correction for the "ICAN moist" atmosphere with the corresponding correction for the "ICAN dry" atmosphere. The differences are due to the presence of water vapour.

/TABLE III

TABLE III

Local height (feet)		50,000				20,000		
Elevation Angle (°)		7°	10°	14°	20°	7°	10°	14°
Refraction correction in feet for	ICAN moist	-605	-320	-168	-92	-150	-80	-39
	ICAN dry	-430	-216	-117	-58	-89	-46	-23

In that refraction of light and radio waves is practically the same for dry air the result for the "ICAN dry" atmosphere has an application in theodolite work.

2. A low level European moisture discontinuity

The error caused by a given angular amount of refraction (Equation (1)) will be greatest when the refraction is concentrated at a single, very low, level. In the European region the effect is greatest in subsidence conditions where warm moist air of sub-tropical origin is present in the lowest layers with dry air overlying. A computation was carried out for two examples of this condition for differing heights of the subsidence layer.

The extreme condition of a constant mixing ratio of 15 g/Kg and temperature of 68°F extending from surface to 1,500 feet, with dry air superimposed, was selected. The true correction at a height of 50,000 feet with an elevation angle of 20° was found to be -92 feet whereas the corresponding "ICAN moist" correction was -90 feet.

The second example was for a change from moist to dry air at 500 feet and otherwise similar conditions. The true correction was found to be -102 feet compared with the "ICAN moist" correction of -90 feet.

The assumption of wholly dry air above the discontinuity is slightly unrealistic. An examination of examples has shown that the effect of small quantities of water vapour in this region will be to reduce the true correction required, as the effect of reduced refraction at a low discontinuity more than balances the increased refraction above due to the increased moisture lapse above. It is possible, therefore, on a given occasion, for the "standard" value either to under-correct within the order of the limit indicated above or to over-correct within the limits of Table III. Reference to M.O.555 part 1 suggests that during the years 1948 to 1950 over Larkhill excess corrections of more than half this range were rarely encountered.

3. A low level sub-tropical discontinuity

The extreme case of high moisture content in the surface layers followed by an abrupt decrease to normal values aloft is a regular summertime feature at Bahrain where dry desert air flows out over the Persian Gulf. Table IV below gives values obtained from a summer-time ascent selected at random.

TABLE IV

Pressure (mb)	996	985	972	960	950	900	850	800
Temperature (°F)	95	97	99	100	99	93	85	77
Mixing Ratio (g/Kg)	29.6	21.4	5.7	5.9	6.2	6.2	5.3	5.0
$(\mu - 1) \times 10^6$	446	387	283	280	279	269	252	239

/It

It will be noted that above 972 mb the μ structure closely approximates to European conditions. The true correction at 50,000 feet with an elevation of 20° was found to be -110 feet compared with an "ICAN moist" correction of -93 feet.

Conclusion

The G.L.III equipment used by the Meteorological Office for wind-finding purposes gives heights which are in error due to refraction approximately by the magnitudes of Fig.2.

Heights are used, as a routine, for assigning mean winds measured over layers of 2,500 to 3,500 feet to standard height levels. In this application the effect of refraction appears insignificant.

The equipment is used also, by comparison with heights computed by radio-sonde data, to assess the systematic errors of the radio-sonde. The probable error at 50,000 feet and elevation 20° of an individual radar height determination is 250 feet and it is necessary to use averages. It is apparent that refraction introduces a systematic error into such differences and that correction for this effect is necessary before radio-sonde performance can be assessed.

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Reference

1. Stickland, A.C. - "Refraction in the lower atmosphere and its applications to the propagation of radio waves" at a conference of the Physical Society and the Royal Meteorological Society - April, 1946.

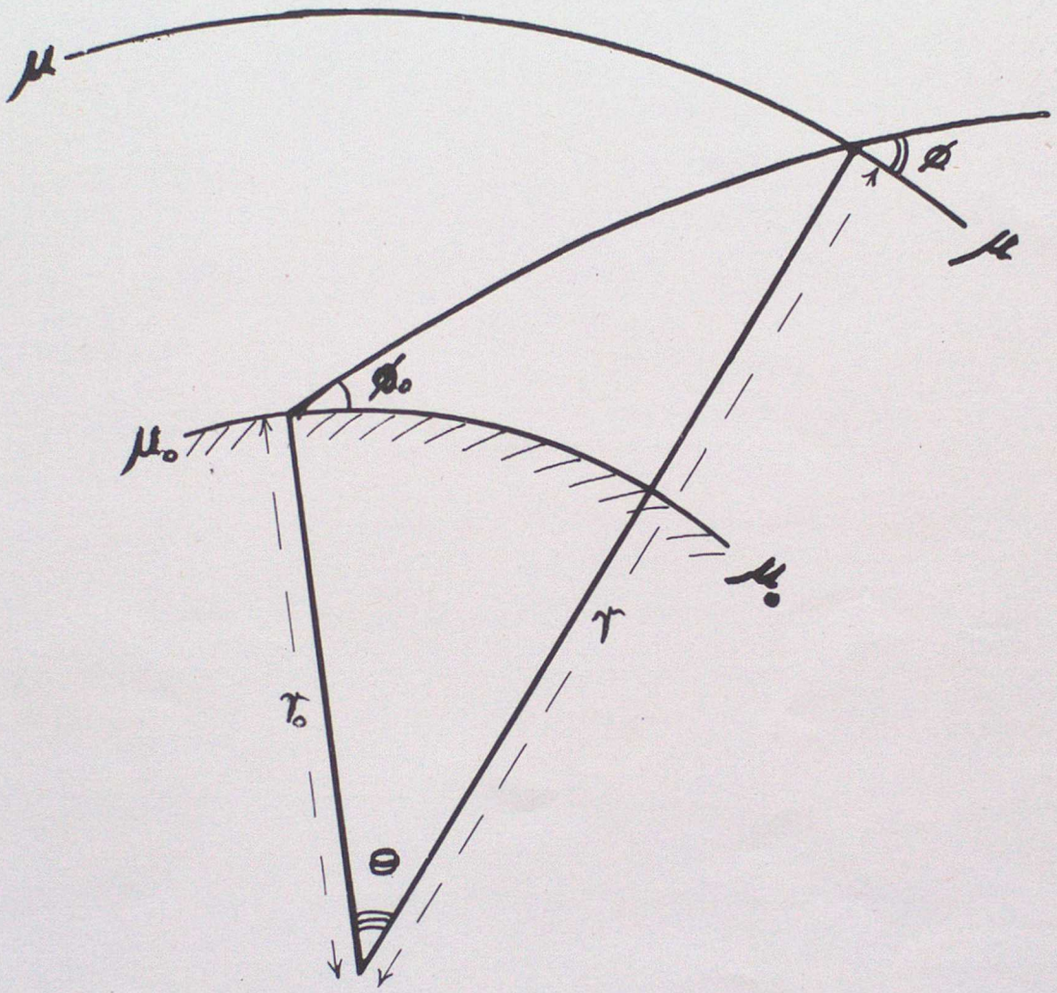
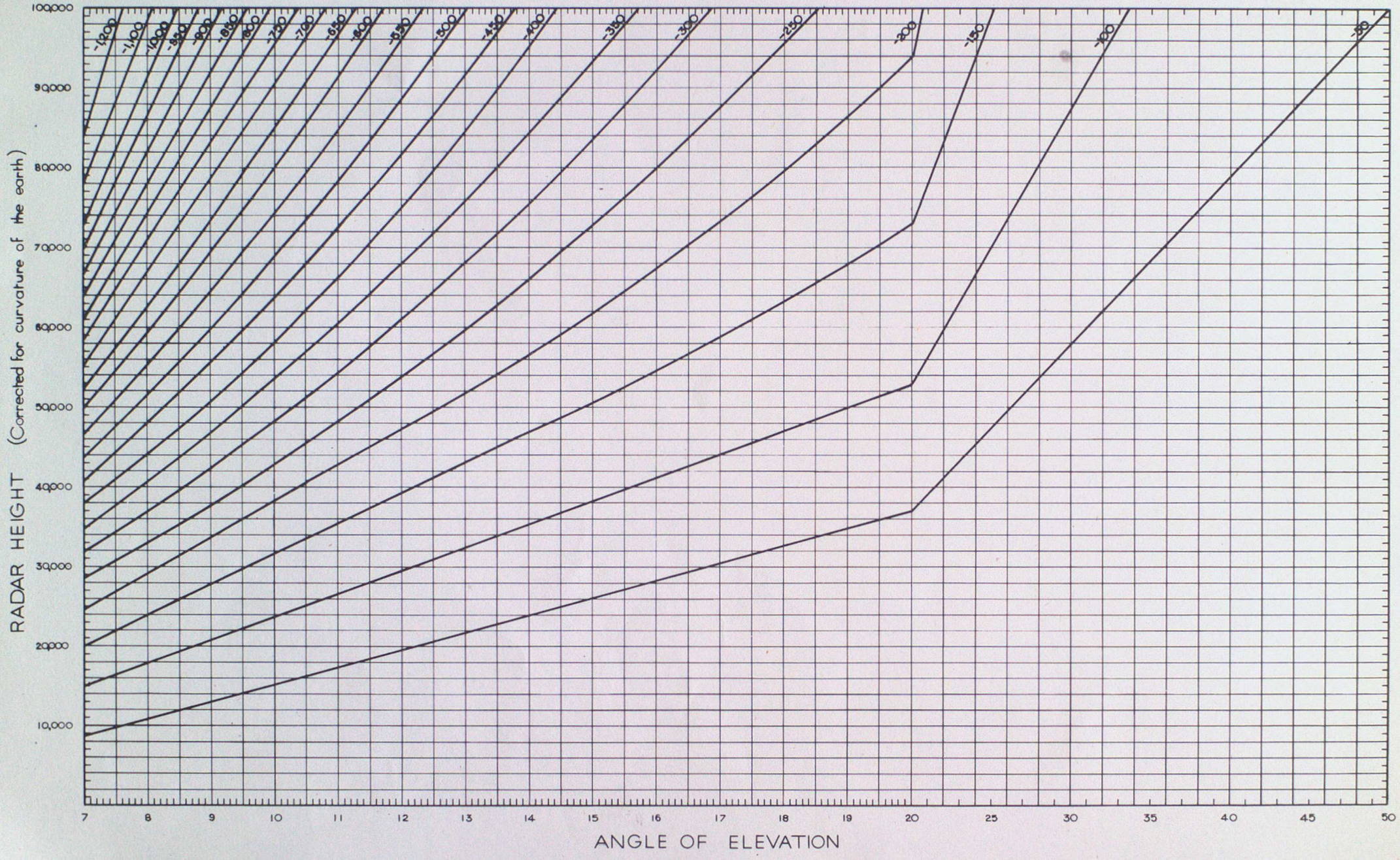


FIG. 1.

FIG.2.

CORRECTION FOR REFRACTION



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